THE DENSITY OF LAWRENCE-KRAMMER AND

NON-CONJUGATE BRAID REPRESENTATIONS OF LINKS

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Abstract. We use some Lie group theory and Budney's unitarization of the Lawrence-Krammer representation, to prove that for generic parameters of definite form the image of the representation (also on certain types of subgroups) is dense in the unitary group. This implies that, except possibly for closures of full-twist braids, all links have infinitely many conjugacy classes of braid representations on any non-minimal number of (and at least 4) strands.

1. Introduction

The Lawrence-Krammer representation [15, 13, 3] ρ_n of the braid group B_n into $SL(p, \mathbb{Z}[t^{\pm 1} q^{\pm 1}])$, with $p = n(n-1)/2$, has become recently of interest as the first faithful representation of braid groups. In this paper, we are concerned with the identification of the image of ρ_n on B_n and certain types of subgroups thereof. An important property, unitarizability, is found by Budney [6] (see theorem 3 below). Our main result is the following.

Theorem 1 Assume q, t with $|t| = |q| = 1$ are chosen so that $t^a q^b = 1$ for $a, b \in \mathbb{Z}$ implies that $a = b = 0$, and the Budney form is definite at q, t. Moreover, assume that ρ_n is irreducible at q, t. Then $\overline{\rho_n(B_n)} = U(p)$ (for $p = n(n-1)/2$).

This is analogous to a previous result for the *Burau representation* ψ_n (which we simply call 'Burau' below) in [24].

The irreducibility of ρ_n will be treated extra with lemma 2. It should be pointed out that it has been proved at separate places. There is written account by M. Zinno [26], though it was observed also by others, incl. V. Jones, R. Budney and W. T. Song. W. T. Song has proved the stronger statement that Budney's form is the only unitarizing form. (He informed me of this result prior to my proof of theorem 1, although now the theorem implies this uniqueness property, at least for definite form, by remark 1.) But it appears all this material is (yet) unpublished. There is closely related work of I. Marin [18], which we discuss in §8. The proof of lemma 2 is provided for completeness and because of its simplicity compared to other methods.

Our main motivation was again the study of braid representations of links. The problem to determine conjugacy classes of braid representations of a given link goes back to the 60s. For some early work see e.g. [21]. With the increasing attention given to braids the problem was studied later e.g. in [8, 19, 9]. We apply theorem 1 to prove

Theorem 2 Assume L is a link and $n > b(L)$. Then there exist infinitely many conjugacy classes of n-braid representations of L, except if

(a) $n \leq 3$ or (possibly)

(b) L is a $(n-1, k(n-1))$-torus link ($k \in \mathbb{Z}$). (This includes the case $k = 0$ of the $n-1$-component trivial link.)

The number $b(L)$ is the minimal number of strands of a braid representation of L, and is called *braid index* (see e.g. [20]). The case (a) is very well-known from [5] to need to be excluded, but we do not know anything about whether any link of case (b) is indeed exceptional. Still the theorem almost completely settles the (in)finiteness for $n > b(L)$. For $n = b(L)$ the situation is far more complicated; there are certain links also for $n \geq 4$ with a single conjugacy class, e.g. unlinks [4], and a further example due to Ko and Lee. Contrarily, Shinjo has announced to me (in preparation) that she has, as extension of her work [22], found very general families of knots with infinitely many minimal braid conjugacy classes. It is possible that the decision problem when finitely many and when infinitely many classes occur for $n = b(L)$ is too complex to have a meaningful answer.

Most of the rest of the paper, until the end of §7, will be devoted to the proof of theorem 1. The proof is rather Lie-group theoretic, and we will need to bring up some related material along the way. In §8 we extend theorem 1 to denseness of the image of subgroups of B_n. In §9 we discuss theorem 2.

2.   Lawrence-Krammer representation and its unitarization

The n-strand *braid group* B_n is considered generated by the Artin *standard generators* σ_i for $i = 1, \ldots, n-1$ [1, 11]. These are subject to relations of the type $[\sigma_i, \sigma_j] = 1$ for $|i - j| > 1$, which we call *commutativity relations* (the bracket denotes the commutator) and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, which we call *Yang-Baxter* (or shortly YB) relations. We write $[\beta]$ for the *exponent sum* of β, and set $B_{k,l} \subset B_n$ for $1 \leq k < l \leq n$ to be the subgroup $< \sigma_k, \ldots, \sigma_{l-1} >$ (where angle brackets mean 'generated by').

The representation ρ_n of B_n can be defined as operating on a complex vector space $R = \mathbb{C}^p$ with $p = n(n-1)/2$ with basis $\{v_{i,j} : 1 \leq i < j \leq n\}$ by

$$\rho_n(\sigma_i)v_{j,k} = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1-q)v_{j,k} & i = j-1, \\ v_{j+1,k} & i = j \neq k-1, \\ qv_{j,i} + (1-q)v_{j,k} - (q^2 - q)tv_{i,k} & i = k-1 \neq j, \\ v_{j,k+1} & i = k, \\ -tq^2 v_{j,k} & i = j = k-1. \end{cases} \tag{1}$$

Here t and q may *a priori* be arbitrary non-zero complex numbers. However, we will choose them always so that $|t| = |q| = 1$. (We will sometimes write q, t explicitly as parameters of ρ_n, with the understanding that a braid cannot be confused with a complex number.) The reason is the following result, which is of main importance below.

Theorem 3 (Budney [6]) The Lawrence-Krammer representation unitarizable if $|q| = |t| = 1$.

In other words, for such t and q, Budney [6] defines a *unitarizing* form $< ., . >$ of ρ_n on $\mathbb{C}^p$. This is a sesquilinear pairing respected by the action of ρ_n, in the sense that for all $\beta \in B_n$ and $x, y \in \mathbb{C}^p$, we have

$$< \rho_n(\beta)x, \rho_n(\beta)y > = < x, y > .$$

This feature is analogous to the form of Squier [23] for the Burau representation ψ_n.

The *(reduced) Burau representation* ψ_n of B_n, depending on a complex parameter q and acting on $\mathbb{C}^{n-1}$, is given by:

$$\psi_n(\sigma_i) = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & 0 & \\ & & 1 & & & & & & \\ & & & 1 & -q & & & & \\ & & & & -q & & & & \\ & & & & -1 & 1 & & & \\ & & & & & & 1 & & \\ & 0 & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix} \quad \text{for } 1 < i < n-1,$$

$$\psi_n(\sigma_1) = \begin{bmatrix} -q & 0 & & & \\ -1 & 1 & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad \text{and} \quad \psi_n(\sigma_{n-1}) = \begin{bmatrix} 1 & & & 0 & \\ & \ddots & & & \\ & & 1 & & \\ 0 & & & 1 & -q \\ & & & 0 & -q \end{bmatrix},$$

where at position (i,i) there is always the entry $-q$.

We used Squier's form previously to carry out a study of the image of ψ_n in [24]. Again Budney's form is definite for proper q and t. We became aware of Budney's result only recently, and so we tried to adapt details we had worked out for ψ_n.

At least one serious obstacle was visible in advance. Due to the quadratic increase in dimension, an argument via a rank estimate for a simple Lie group is more complicated. Still most simple Lie group representations have a dimension larger than quadratic (in the rank), and with a certain amount of extra effort we will be able to deal with them.

Remark 1 Let us here remark that when $< .,.>$ is definite, the subgroup of linear transformations of $\mathbb{C}^n$ respecting the form is conjugate in $SL(n,\mathbb{C})$ to $U(n)$. Conversely, each such subgroup determines the respected pairing up to complex conjugation and multiples. This follows from that facts that the only outer automorphism of $SU(n)$ is (entry-wise) complex conjugation, and the centralizer of $SU(n)$ in $SL(n,\mathbb{C})$ are the scalar matrices.

It will not be necessary to study the form very explicitly here. We need only the following consequence of the formula in the proof of Theorem 3 of [6]. Below for a complex number $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in (-\pi, \pi]$ we set $|z| = r$ and $\arg z = \theta$ (when $r \neq 0$).

Proposition 1 If $q_i, -t_i \to 1$ with $|q_i| = |t_i| = 1$ are chosen so that $\dfrac{1 - q_i}{1 - q_i\sqrt{-t_i}} \to 0$, then the Budney form is definite at q_i, t_i for i large.

Remark 2 The additional condition means that $\arg q_i / \arg -t_i \to 0$. In other words, when t_i is close to -1, one should choose q_i close to 1 in a way depending on t_i. It is clear that one can choose such q_i, t_i which are algebraically independent. This property will be used in the proof of theorem 2, but it will *not* be relevant before §9.

The reason why we are interested in the value $t = -1$ is (see [13]):

Lemma 1 ρ_n turns into the symmetric square of ψ_n for $t = -1$.

Then, for $q = 1$ we have (the symmetric square of) the permutation homomorphism π_n. The homomorphism π_n means here the $n - 1$-dimensional (irreducible) representation obtained from the action of the symmetric group S_n (onto which there is an obvious homomorphism from B_n) on the coordinates of $\mathbb{C}^n$, after removing the (invariant) space generated by $(1, 1, \ldots, 1)$. The notion of symmetric square is explained more precisely below.

Warning 1 The following should be kept in mind regarding the t variable.

1. The convention of [13] for the matrices of the representation differs from [2, 6]; t of former is $-t$ of latter. We stick with Bigelow-Budney's convention for t.

2. Also, in theorem 4.1, p.483 of [2] there is a misprint: in the fourth option t should be $-t$; this is set right on p.782 of [6], and our (1) is a reproduction of latter formula.

3. Next, t is used often, e.g. in [12], as the variable for ψ_n, but it is here q, *not* t, that via lemma 1 originates from Burau (and what we may call the Burau variable). Apart from replacing t by q, our definition of ψ_n is as in [12].

Let us clarify and fix some language. For a (complex) vector space V with basis $e_1, \ldots, e_n$, we can define the *symmetric square* $\mathrm{Sym}^2 V$ to be subspace of $V \otimes V$ spanned by elements

$$v \odot w = \frac{v \otimes w + w \otimes v}{2}.$$

$\mathrm{Sym}^2 V$ has the standard basis

$$\{ e_i \odot e_j \ : \ i \leq j \}.$$

When an endomorphism f acts on a vector space V, then it induces an endomorphism on $\mathrm{Sym}^2 V$ we write $\mathrm{Sym}^2 f$. So we can talk of $\mathrm{Sym}^2 \psi_n$.

We will also need the *antisymmetric square* $\Lambda^2 V$ generated by

$$v \wedge w = \frac{v \otimes w - w \otimes v}{2}.$$

Similarly there is a meaning to $\bigwedge^2 f$.

Lemma 2 Assume q, t with $|t| = |q| = 1$ are chosen so that $t \neq \pm 1$ and q is, dependingly on t, sufficiently close to 1. Also, assume the Budney form is definite at q, t. Then ρ_n is irreducible at q, t.

Proof. For $(q, t) = (1, -1)$ the symmetric square $\mathrm{Sym}^2 \pi_n$ of the permutation homomorphism acts by permuting the indices in $v_{i,j}$. Now, when $q = 1$, but $t \neq \pm 1$, then the action is similar, except that σ_i acts on $v_{i,i+1}$ by multiplying by $-t$. It is clear then that such an endomorphism has eigenvalues ± 1 and $-t$, with the eigenspace for $-t$ being 1-dimensional, generated by $v_{i,i+1}$.

Since by unitarity every invariant subspace has an invariant (orthogonal) complement, it follows that if there is an invariant subspace V, then $V^\perp$ is also invariant, and one of both, say w.l.o.g. V, contains $v_{1,2}$. But then by the way the action is described, $V = R$.

Thus ρ_n is irreducible for $q = 1$, and then for q close to 1 because irreducibility is an open condition. $\square$

Remark 3 The reason we propose this proof, apart from its simplicity, is to outline a way in which the algebraic independence condition on q, t can be circumvented. This condition inavoidably enters if one likes to return from formal algebra to complex-valued q, t, as in the approaches of [18] and [26], where in latter paper, irreducibility follows from the identification of ρ_n to a summand of the BWM algebra. Algebraic independence is 'generically' satisfied, but for many concrete values of q, t it may be false, or at least difficult to establish. (It is needed for the faithfulness, but latter is not essential in our arguments, until §9.) It is clear from our proofs that the condition 'q close to 1' can, in both lemma 1 and 2, be made precise by a slightly more technical calculation. The clarification for which parameters *exactly* ρ_n is irreducible requires far more effort, and is only subject of ongoing work. The sole written reference I received, only *a posteriori*, from I. Marin, is a very recent Ph D thesis of C. Levaillant [16]. (Allegedly Bigelow has an own, unpublished, proof.)

3. Lie groups

3.1. Correspondence between compact and complex Lie groups

We start by reviewing a few basic facts from Lie group theory, which mostly occurred in our treatment of Burau [24].

Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Compactness implies in particular that G is real, finite-dimensional and linear reductive. Linear reductive means for a closed subgroup $G \subset GL(n, \mathbb{C})$ that the number of connected components is finite and G is closed under conjugated matrix transposition $M \mapsto \overline{M^T}$.

A *linear representation* of G is understood as a pair $\rho = (V, \pi)$ made of a vector space V and a homomorphism $\pi : G \to \mathrm{Aut}(V)$. We will often omit π and identify ρ with V for simplicity, if unambiguous. A representation is *irreducible* (and will be often called *irrep* below) if it has no non-trivial (i.e. proper and non-zero) invariant subspaces. Linear reductiveness of G implies that each invariant subspace of a linear representation of G has a complementary invariant subspace, so that each representation of G is *completely reducible* as direct sum of irreducible representations.

To G there exists a uniquely determined complex connected linear reductive Lie group $G_{\mathbb{C}}$, with

(i) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is the Lie algebra of $G_{\mathbb{C}}$, and
(ii) $G \subset G_{\mathbb{C}}$ as a closed subgroup.

Then $G_{\mathbb{C}}$ is called a *complexification* of G. If G is simply connected, so is $G_{\mathbb{C}}$, and then any other connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is a covering of $G_{\mathbb{C}}$. The real group G is always a maximal compact subgroup of $G_{\mathbb{C}}$; we call it the *compact real form* of $G_{\mathbb{C}}$.

Thus we have a one-to-one correspondence between a compact connected real (simply-connected) Lie group and a (simply-connected) connected linear reductive complex Lie group. Under this correspondence to $G = SU(n)$ we have $G_{\mathbb{C}} = SL(n, \mathbb{C})$. The groups are connected and simply-connected (for $n \geq 2$).

The correspondence behaves well w.r.t. many properties. The real form G is simple, if and only if $G_{\mathbb{C}}$ is too. (In particular, if G is semisimple, so is $G_{\mathbb{C}}$.) For every complex representation $\rho = (V, \pi)$ of G ('complex' means that V is a complex vector space) we have an 'extension' to a representation $\tilde{\rho} = (V, \pi_{\mathbb{C}})$ of $G_{\mathbb{C}}$, such that $\pi_{\mathbb{C}}$ is an extension of π from G to $G_{\mathbb{C}}$. This extension is still faithful if G is compact.

3.2. Symmetric pairs

Let G be a Lie group and σ an involution. Define

$$G^{\sigma} := \{ g \in G : \sigma(g) = g \}$$

to be the σ-invariant subgroup of G and G_0^{σ} the connected component of the identity. Then a pair (G, H) for a closed subgroup H with $G_0^{\sigma} \subset H \subset G^{\sigma}$ is called a *symmetric pair*.

In the case $G = SU(n)$ the symmetric pairs have been classified by Cartan. See [14, Chapter IX.4.A, table p.354]. In this case H is some of $S(U(m) \times U(n-m))$, $Sp(n/2)$ if n is even, or $SO(n)$.

Let us give the corresponding involutions σ that define the symmetric pairs (see p. 348 of [14]).

Define $M_{i,j}$ to be the matrix with all entries 0 except that at the (i, j)-position, which is 1. Let $\mathrm{diag}(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i M_{i,i}$ be the diagonal matrix with entries $x_1, \ldots, x_n$, so that $Id_n = \mathrm{diag}(1, \ldots, 1)$ (with n entries '1') is the identity matrix.

For $S(U(m) \times U(l))$, with $m + l = n$, the involution σ is of the form $\sigma_{m,l} : M \mapsto I_{m,l} M I_{m,l}$, where

$$I_{m,l} = \mathrm{diag}(\underset{m}{1}, \ldots, 1, \underset{m+1}{-1}, \ldots, -1).$$

For $n = 2n'$ even, $Sp(n')$ respects the involution $\sigma_J : M \mapsto J^{-1} \bar{M} J$, where

$$J = \left[\begin{array}{c|c} 0 & -Id_{n'} \\ \hline Id_{n'} & 0 \end{array} \right], \tag{2}$$

and $\bar{M}$ is the complex conjugation (of all entries) of M. For $SO(n)$, the involution σ is given by $\sigma(M) = \bar{M}$.

These subgroups can be also defined in the standard representation by the linear transformations that respect a certain (complex) non-degenerate bilinear form, which is Hermitian, skew-symmetric or symmetric resp. All transformations that respect such a form determine, up to conjugacy, a subgroup of one of the three types.

Analogous three types of subgroups $R(m,n,\mathbb{C})$, $Sp(n/2,\mathbb{C})$ and $SO(n,\mathbb{C})$ can be defined for $SL(n,\mathbb{C})$. Here $R(m,n,\mathbb{C})$ is the group of all (complex-)linear unit determinant transformations of $\mathbb{C}^n$ that leave invariant a subspace of dimension m. (In contrast to the unitary case, there is not necessarily a complementary invariant subspace!)

We call the three types of groups *reducible*, *symplectic* and *orthogonal* resp. We call a representation V of G after one of the types, if it is contained in a conjugate of a group of the same name. Orthogonal and symplectic subgroups/representations will be called also *symmetric*, the others *asymmetric*.

In the real-complex correspondence we explained, we have that *if the representation ρ of G is symmetric, then so is the representation $\tilde{\rho}$ of $G_{\mathbb{C}}$.* This is easily seen by restricting the respected bilinear form to the reals.

4.   The maximal subgroups

Convention 1 From here, until the conclusion of the proof of theorem 1 at the end of §7, we will assume, unless we clearly indicate otherwise, that q,t are fixed unit norm complex numbers that *satisfy the assumptions of theorem 1*. In this situation, we will usually omit explicit reference to the parameters q,t in the notation. We may mention here that these conditions are stronger than we actually need, and were chosen so as to keep the formulation of theorem 1 simpler. We will elaborate on weaker (but slightly more technical) sufficient conditions in §8.

For our approach to theorem 1, we will mainly study the *normalization* $\rho'_n \subset SU(p)$. That is, we consider the (determinant) factorization $U(p) = SU(p) \times U(1)$ and the projection on the first factor. This means that we multiply ρ_n by a power of a scalar $\mu \in \mathbb{C}$ (depending on q,t),

$$\rho'(\beta) = \mu^{[\beta]} \cdot \rho_n(\beta),$$

so that $\rho'(\beta)$ has determinant 1. Here $[\beta]$ is the exponent sum of β, its image under the homomorphism $B_n \to \mathbb{Z}$ sending all σ_i to 1. This scalar μ can be calculated as in (6).

We will prove that

$$\overline{\rho'_n(B_n)} = SU(p). \tag{3}$$

If then $\overline{\rho_n(B_n)} \neq U(p)$, it must have codimension one, and so be a collection of components isomorphic to $SU(p)$. But we have

$$\det(\rho_n(\sigma_i)) = -t(-q)^n, \tag{4}$$

which is not a root of unity by assumption, so the projection of $\overline{\rho_n(B_n)}$ onto $U(1)$ is not discrete.

To prove (3) we argue indirectly, and assume the contrary.

Since $\hat{H} = \overline{\rho'_n(B_n)}$ is a compact Lie subgroup of $SU(p)$, we can complexify its connected component $\hat{H}_0$ of the identity, and obtain a (faithful) representation of a reductive complex Lie subgroup $\tilde{H}$ of $SL(p,\mathbb{C})$, which is a proper subgroup by dimension reasons. It is contained in a maximal proper subgroup which we call H. We will show that $H = SL(p,\mathbb{C})$, and have a contradiction.

Convention 2 Let us stipulate that until the end of the proof of lemma 4, we use n to indicate the dimension rather than the number of braid strands (ρ_n will not appear in this scope).

In the 1950s, Dynkin published a series of seminal papers in the theory of Lie groups. One of his remarkable achievements was the classification of maximal subgroups of classical Lie groups [7]. (See theorems 1.3, 1.5 and 2.1 in [7].)

Theorem 4 (Dynkin [7]) A maximal proper subgroup of $SL(n,\mathbb{C})$ is conjugate in $SL(n,\mathbb{C})$ to
(i)   some symmetric representation, i.e., $SO(n,\mathbb{C})$ or $Sp(n/2,\mathbb{C})$ when n is even, *or*
(ii) to $SL(m,\mathbb{C}) \otimes SL(m',\mathbb{C})$ with $mm' = n$ and $m,m' \geq 2$ (one which is non-simple irreducible), *or*
(iii) to $R(m,n,\mathbb{C})$ (one which is reducible), *or*
(iv) it is an irreducible representation of a simple Lie group.

For a non-simple group $H = H_1 \times H_2$, one considers $\mathbb{C}^n = \mathbb{C}^{mm'} \simeq \mathbb{C}^m \otimes \mathbb{C}^{m'}$ as a tensor (Kronecker) product, and H_1 resp. H_2 acts on $\mathbb{C}^m$ resp. $\mathbb{C}^{m'}$.

We call the first 3 types of subgroups *orthogonal*, *symplectic*, *product* and *reducible* resp. We will exclude these types in the next section, before we get to deal with (iv).

5.   Excluding symmetric, product and reducible representations

Note that case (i) in theorem 4 is in fact included in case (iv). It is singled out because it can be handled by a more elementary eigenvalue analysis. What theorem 4 actually achieves is also a description of the maximal subgroups of $SO(n,\mathbb{C})$ and $Sp(n/2,\mathbb{C})$, and one can decide which of the representations of case (iv) are symmetric.

It will be useful to have an elementary analysis of the eigenvalues on the various subgroups.

Proposition 2 If $\lambda \in \mathbb{C}$ is an eigenvalue of an orthogonal or symplectic matrix M in $SU(n)$, then so is $\bar{\lambda}$.

Proof. Orthogonal matrices are conjugate to real ones, so their characteristic polynomial has real coefficients. So consider a symplectic matrix M. Then the operator J in (2) of the involution σ_J that M respects (up to conjugacy) satisfies $J^2 = -Id_n$. So $M = J^{-1}\bar{M}J$ is equivalent to $MJ = J\bar{M}$. Now by assumption there is a vector $v \in \mathbb{C}^n \setminus \{0\}$ with $Mv = \lambda v$. Then

$$MJ\bar{v} = J\bar{M}\bar{v} = J\overline{Mv} = J\overline{\lambda v} = \bar{\lambda}J\bar{v},$$

so $J\bar{v} \neq 0$ is an eigenvector of M for eigenvalue $\bar{\lambda}$.   □

Lemma 3 Assume that $H = H_1 \otimes H_2 \in SL(n_1,\mathbb{C}) \otimes SL(n_2,\mathbb{C})$. Let λ_i be the eigenvalues of H_1 (counting multiplicities) and μ_j those of H_2. Then the eigenvalues (counting multiplicities) of H are $\lambda_i\mu_j$.

Proof. $H_{1,2}$ have by Jordan box decomposition lower-triangular bases e_i and f_j. Order the tensor basis $\{e_i \otimes f_j\}$ so that elements with smaller $i+j$ appear first. Then w.r.t. the so ordered basis, $H = H_1 \otimes H_2$ is lower-triangular. Since in lower-triangular matrices, the eigenvalues appear on the diagonal, the claim is clear.   □

Lemma 4 Assume that for $H = H_1 \otimes H_2 \in U(n_1) \otimes U(n_2)$ all eigenvalues λ_i except exactly one (and single-multiplicity) eigenvalue λ_{i_0} satisfy $\lambda_i \in \{1,a\}$ for $a \in \mathbb{C}$ with $a \neq \pm 1$. Furthermore assume that
(i) $\lambda_{i_0} \neq a^2$ and that
(ii) $\lambda_{i_0} \neq 1/a$, or $\lambda_i = 1$ for at least half of all λ_i.
Then one of n_1 or n_2 is equal to 1, in other words, H is a direct product only in a trivial way.

Proof. Let H_1 have eigenvalues μ_j and H_2 have eigenvalues ν_k. Since $H_{1,2}$ are diagonalizable, H has eigenvalues $\lambda_i = \mu_j\nu_k$, with all $\mu_j,\nu_k \neq 0$.

Let $\lambda_{i_0} = \mu_{j_0}\nu_{k_0}$ and assume $n_1,n_2 > 1$, where n_1 is now the number of μ_j and n_2 that of ν_k. By choosing $k \neq k_0$ and looking at the set $\{\mu_j\nu_k\}$ (for the fixed k but varying j), we see that $\{\mu_j\} \subset \{x,y\}$ and $x/y = a^{\pm 1}$. A similar conclusion applies to $\{\nu_k\}$ using μ_j for some fixed $j \neq j_0$.

Then it is clear that $\lambda_{i_0} \in \{a^2, 1/a\}$, and we assumed that the former value is not taken. If $\lambda_{i_0} = 1/a$, then (for $a \neq \pm 1$) it follows that exactly one of the μ_j is different from all the others, which are equal, and similarly for ν_k. This implies that the multiplicity of $\lambda_i = a$ is $(n_1-1)(n_2-1) \geq n/2 - 1$, for the number $n = n_1n_2$ of eigenvalues λ_i (with $n_1,n_2 > 1$). Then, $\lambda_i = 1$ occurs $n - (n_1-1)(n_2-1) - 1 \leq n/2$ times, which we excluded. This gives a contradiction to the assumption $n_1,n_2 > 1$.   □

Now we apply the preceding lemmas to ρ_n (with n resuming the meaning of number of braid strands).

Lemma 5 The image of ρ'_n for $n \geq 3$ is not orthogonal or symplectic, and hence neither is H.

Proof. Assume first that $\rho'_n(\sigma_1) \in \hat{H}_0$, in the notation of §4. The eigenvalues of $\rho_n(\sigma_1)$ can be easily determined from the Krammer matrix, and replacing t by $-t$ according to the warning 1. The result is

$$\left\{ -tq^2, \ \underbrace{-q,\ldots,-q}_{n-2 \text{ times}}, \ \underbrace{1,\ 1,\ \ldots,1}_{(n-1)(n-2)/2 \text{ times}} \right\}. \tag{5}$$

 Let us fix, also for outside this proof, the following notation. A set of n copies of k will be written $\{k\}^n$, and union of such sets will be written as a product. Then the above set can be written as $\{-tq^2\}\{-q\}^{n-2}\{1\}^{(n-1)(n-2)/2}$.

To normalize for the eigenvalues of $\rho'_n(\sigma_1)$, this set has to be multiplied by

$$\mu = \det(\rho_n(\sigma_1))^{-2/n(n-1)}, \tag{6}$$

 with (4). For the chosen q,t, none of the resulting numbers is real (i.e. ± 1). This in particular finishes the cases $n(n-1)/2$ odd, so $n \geq 4$. But then there is a pre-dominant occurrence of μ, and the set is not closed under conjugation.

Now let $\rho'_n(\sigma_1) \notin \hat{H}_0$. By the assumption on q,t in theorem 1, no two distinct eigenvalues λ_i of $\rho_n(\sigma_1)$ have a quotient which is a root of unity. Then one can choose a number m large enough so that all eigenvalues of $\rho_n(\sigma_1^m)$ are as close to 1 as desired. (This can be seen for example by looking at the closure of the infinite cyclic subgroup generated by $(\lambda_1, \lambda_2, \lambda_3)$ for the 3 distinct λ_i within the 3-dimensional torus T^3, and arguing that this closure, which is formally a Lie subgroup of T^3, cannot have a codimension.) Therefore, by compactness $\rho'_n(\sigma_1^m) \in \hat{H}_0$. Then one argues analogously to above with $\rho'_n(\sigma_1^m)$. $\qquad \square$

Lemma 6 The image of ρ'_n is not contained in a Kronecker product.

Proof. Consider first the situation when $\rho'_n(\sigma_1) \in \hat{H}_0$. We want to show that there are no (non-trivial) matrices H_1, H_2 with

$$\rho'_n(\sigma_1)(q,t) = H_1(q,t) \otimes H_2(q,t). \tag{7}$$

 To rule out (7), we may replace ρ'_n by ρ_n, and consider the eigenvalues of $\rho_n(\sigma_1)$. Using (5), and under the restrictions on q,t of convention 1, we can apply lemma 4. It gives the desired conclusion.

Again, for $\rho'_n(\sigma_1) \notin \hat{H}_0$, one argues with σ_1^m and replaces t by t^m (to which the restriction of theorem 1 applies in the same way). $\qquad \square$

Lemma 7 The group H acts irreducibly on $\mathbb{C}^p$.

Proof. Since, in the notation of §4, we have $H \supset \tilde{H} \supset \hat{H}_0$, it is enough to prove that irreducibility of ρ'_n, or simpler of ρ_n, is not spoiled when we restrict $\hat{H}$ to $\hat{H}_0$.

When the Budney form is definite, each ρ_n-matrix diagonalizes. Thus invariant subspaces and irreducibility will be preserved if we pass to m-th powers of any generating set $\{\tau_i\}$ of B_n, provided there are no two distinct eigenvalues of any $\rho_n(\tau_i)$ which differ (multiplicatively) by a root of unity.

By the assumption on q,t in theorem 1, this condition holds for $\tau_i = \sigma_i$ from (5). Clearly one can choose m in the last paragraph of the proof of lemma 5 so that the eigenvalues of all $\rho_n(\sigma_i^m)$ for $1 \leq i \leq n-1$ are as close to 1 as desired, and then all $\rho'_n(\sigma_i^m) \in \hat{H}_0$. $\qquad \square$

6. Rank estimate

In the quest for what H could be, we are left from the list of theorem 4 only with case (iv). To deal with this, in the following it is necessary to appeal to a larger extent to the Lie theory described, mainly in the appendix, in [7]. We will repeat a certain part, though we would have to refer there for further details.

Convention 3 References to pages, and to equations or statements numbered '0. $\cdots$' are to be understood *to Dynkin's paper* (in the translated version).

The *rank* $\operatorname{rk} G$ of a simple Lie group G is the maximal dimension of a torus G contains, or the number of nodes in its Dynkin diagram. The latter description will be used from the next section on. Here we have to deal with the torus.

We will recur our rank estimate for ρ_n to the one for ψ_n by means of the important observation in lemma 1.

Let β be a *fixed* braid in B_n. A braid $\beta \in B_k$ for $k \le n$ can be regarded also as a braid $\beta \in B_{1,k} \subset B_n$. The following lemma tells us how to determine the eigenvalues of $\rho_n(\beta)$.

Let for a matrix M, by $E = \operatorname{Ev} M = \{\lambda_i\}$ be denoted the eigenvalues of M (counting multiplicities), and let $\operatorname{Sym}^2 E = \{\lambda_i \lambda_j : i \le j\}$.

Lemma 8

$$\operatorname{Ev} \rho_n(\beta) = \big(\operatorname{Sym}^2 \operatorname{Ev} \psi_n(\beta) \setminus \operatorname{Sym}^2 \operatorname{Ev} \psi_k(\beta) \big) \cup \operatorname{Ev} \rho_k(\beta).$$

For q, t of definite Budney form, the eigenspaces of $\rho_n(\beta)$ of eigenvectors in $\operatorname{Ev} \rho_k(\beta)$ correspond to $\mathcal{E}_k := \{ v_{ij} : 1 \le i < j \le k \}$.

Proof. We order the basis $\mathcal{E}_n$ of $\mathbb{C}^p$ so that $\mathcal{E}_k$ occur first. It is obvious from the definition (1) that $\rho_n\big|_{B_k}$ respects $\mathcal{E}_k \subset \mathcal{E}_n$. So the matrix of $\beta \in B_k \subset B_n$ has the form

$$\rho_n(\beta) = \left[\begin{array}{c|c} \rho_k(\beta) & 0 \\ \hline A & B \end{array} \right], \tag{8}$$

where A, B also depend on β. Thus

$$\operatorname{Ev} \rho_n(\beta) = \operatorname{Ev} \rho_k(\beta) \cup \operatorname{Ev} B(\beta). \tag{9}$$

The next important observation is that by definition the variable t does not occur in B for $\rho_n(\sigma_i)$, $i = 1 \dots, k-1$. Then the same is also true for their inverses, and finally thus for $\rho_n(B_k)$. But since B does not depend on t, its eigenvalues can be determined setting $t = -1$. Then $\rho_k = \operatorname{Sym}^2 \psi_k$ and $\rho_n = \operatorname{Sym}^2 \psi_n$. Thus we have

$$\operatorname{Ev} B(\beta) \cup \operatorname{Sym}^2 \operatorname{Ev} \psi_k(\beta) = \operatorname{Sym}^2 \operatorname{Ev} \psi_n(\beta). \tag{10}$$

Combining (9) and (10), we have the claim. □

Note that (5) also follows from this lemma.

Now we apply the lemma on the following elements in B_n that were of central importance also for Burau.

$$\beta_{n,k} = \Delta_k^2 = (\sigma_1 \dots \sigma_{k-1})^k \in B_n. \tag{11}$$

Lemma 9 When q, t are chosen as in theorem 1, and the Budney form is definite, we have

$$\overline{\rho_n'(< \beta_{n,2}, \dots, \beta_{n,n-1} >)} = T_{n-2},$$

an $n-2$-dimensional torus. Thus in particular $\operatorname{rk} H \ge n - 2$.

Proof. We have that $\rho_k(\beta_{n,k})$ are scalars, and with the notation explained below (5), we have

$$\mathrm{Ev}\,\psi_n(\beta_k) = \{q^k\}^{k-1}\{1\}^{n-k}\,,$$

as observed in [24]. If the Budney form is definite on B_n, then so it is on B_k, and all matrices are diagonalizable. The eigenspaces corresponding to the eigenvalues $\{q^k\}^{k-1}$ of ψ_n are spun by $e_1 \ldots e_{k-1}$. The claim follows by a careful look at eigenvalues and eigenspaces. $\qquad\square$

7.   The irreps of simple Lie groups

7.1.   Dynkin diagrams. Weyl's dimension formula

With §6 we are left to consider irreps of simple Lie groups. Moreover we know quite exactly the dimension. It will be more convenient to look at our rank estimate in lemma 9 from the point of view of the group, not the number of braid strands. It says then that for a group of rank n, the dimension of the representation must be $(n'+1)(n'+2)/2$ for some $n' \leq n$. Moreover, this irrep should not have an invariant form by §5. We try next to find out how to obtain all these irreps.

By the work of Cartan, irreps ϕ of a simple Lie group G are determined by their *highest weight* Λ, and latter is completely described by the property that

$$a_i = \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \tag{12}$$

are non-negative integers for all simple roots α_i of the Lie algebra $\mathfrak{g}$ of G (and at least one a_i is positive). The scalar product $(.,.)$ is the one defining the Dynkin diagram: with the normalization that (in the cases we require below) all simple roots have length 1, nodes of the Dynkin diagram depicting orthogonal vectors are not connected, and connected nodes correspond to vectors of scalar product $-1/2$.

Since the α_i correspond to nodes in the Dynkin diagram, our convention, as in [7, p.329 top], will be to write a_i at the node for α_i in the diagram (but omit zero entries). We will refer to a_i also as *labels* of the nodes. If $a_i = 0$ for all i except exactly one, where $a_i = 1$, we call ϕ a *basic* representation; it is obviously associated to the simple root (or node) α_i with $a_i = 1$.

The following formula calculates the dimension $N(\phi)$ of the irrep ϕ corresponding to Λ.

Lemma 10 (Weyl's formula, Theorem 0.24)

$$N(\phi) = \prod_{\alpha \in \Sigma_+} \frac{(\Lambda + g, \alpha)}{(g, \alpha)}\,, \tag{13}$$

where Σ_+ is the set of positive roots of $\mathfrak{g}$ and

$$g = \frac{1}{2} \sum_{\beta \in \Sigma_+} \beta\,. \tag{14}$$

Mostly we will appeal to the following consequence. (We have used the fact in a weaker form already for Burau.)

**Lemma 11** Assume that one increases the label a of a node in the Dynkin diagram to $a+1$ ($a \geq 0$). Then the dimension of the irrep grows a least by a factor of $(a+2)/(a+1)$, in particular it increases strictly.

Proof. If α is a simple root, then it is known that

$$\frac{2(g, \alpha)}{(\alpha, \alpha)} = 1\,,$$

(see (0.141)), so that $(g, \alpha) > 0$. Then this is also true for all $\alpha \in \Sigma_+$, since they are just sums of (distinct) simple roots. For the same reason (Λ, α) are just sums of a_i, and so non-negative.

Thus increasing some a_i will not decrease any of the (positive) factors in the product of (13). The estimate of increase of $N(\phi)$ follows from looking at the factor that corresponds to α_i. $\qquad\square$
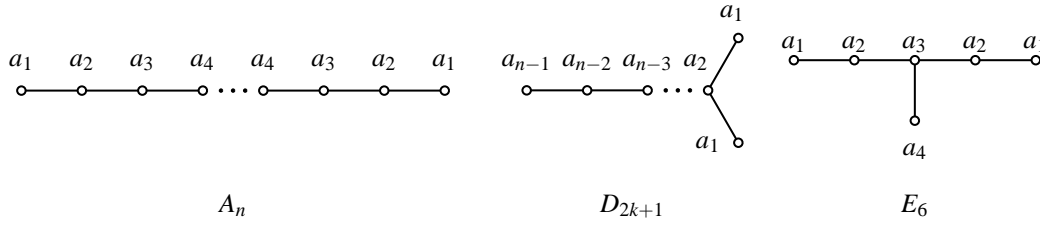
Figure 1: The marking of symmetric irreps (i.e. which admit an invariant symmetric or antisymmetric form).

Definition 1 Let us say that an irrep ϕ' *dominates* another irrep ϕ, if the labels a_i' of ϕ' and a_i of ϕ satisfy $a_i' \geq a_i$ for all i.

A further important tool is the decision which irreps are asymmetric. This goes back to work of Malcev, and can be done as explained in Theorem 0.20 p.336 and Remark C.a. on p.254. With the exclusion of symmetry in §5, we are left only with representations of A_n, D_{2k+1} and E_6, whose labelings do *not* admit a certain symmetry as shown on Figure 1 (which reproduces Table 3, p.365 in [7]).

7.2. A_n

Let π_1 be the elementary representation
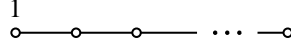
of A_n as $SL(n+1,\mathbb{C})$. Our assumption was that $H \neq \pi_1$, so we will discuss the other possibilities.

First we look at the dimension of the basic representations in Table 30, p.378. These representations were written as π_k in (0.92). For symmetry reasons it makes sense to consider only $n \geq 2k-1$.

If $k \geq 3$, then the only case of

$$N(\pi_k) = \binom{n+1}{k} \leq \frac{(n+1)(n+2)}{2}$$

is that of $k = 3, n = 5$. This representation
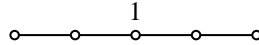
is symmetric, and it has improper dimension 20. So we must consider (to avoid symmetry) representations dominating

  and   (15)

 But latter two dimensions are by lemma 11 at least $2 \cdot 20 = 40 > 21$. (The exact dimensions are 105 and 210, resp.)
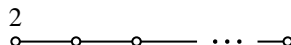
The representations

  and

have dimensions

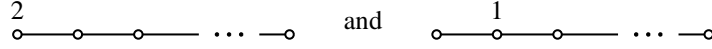$$\frac{(n+2)(n+1)^2 n}{12} \quad \text{and} \quad \frac{(n+2)(n+1)n}{3}.$$

They are too large for $n \geq 3$. The dimension of

is precisely $\frac{(n+1)(n+2)}{2}$, so that the first node label cannot be increased beyond 2.

There remain the irreps

$$\overset{2}{\circ}\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\;\cdots\;-\!\!\circ \qquad \text{and} \qquad \circ\!\!-\!\!-\!\!\overset{1}{\circ}\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\;\cdots\;-\!\!\circ$$

These are $\mathrm{Sym}^2\pi_1$ and $\bigwedge^2\pi_1$. This can be seen as follows: the inclusions are present by p.342 l.-8 in the first case, and p.347 l.11 in the second case. Equality follows by calculating dimensions of the irreps using (0.148).

So it remains to rule out the symmetric and antisymmetric square of the elementary representation π_1. For symmetry reasons $\mathrm{Sym}^2\pi_1$ must be considered for $n \geq 2$, and $\bigwedge^2\pi_1$ for $n \geq 4$ only. By comparing dimensions, it is clear that the number m of strands of ρ_m is $m = n+2 \geq 4$ in the first, and $m = n+1 \geq 5$ in the second case.

We will have to count again eigenvalues, and use the following suggestive

Lemma 12 If M has eigenvalues λ_i (counting multiplicities) then the eigenvalues of $\mathrm{Sym}^2 M$ are $\{\lambda_i\lambda_j : i \leq j\}$ and those of $\bigwedge^2 M$ are $\{\lambda_i\lambda_j : i < j\}$.

Proof. Similar to lemma 3 (only this time with bases $e_i \odot e_j$ resp. $e_i \wedge e_j$). □

We can thus finish the A_n series with the following

Lemma 13 $\rho'_n(\sigma_1^m)$ is not a symmetric square for $n \geq 4$ and not an antisymmetric square for $n \geq 5$.

Proof. It is clear that one can argue with ρ_n instead. Consider again $q = -1$. Then all eigenvalues but exactly one are 1. Let λ_i be the ($n-1$ resp. n) eigenvalues of a potential matrix, whose (anti)symmetric square is $\rho_n(\sigma_1)$.
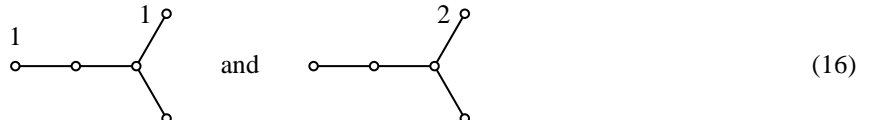
For symmetric square $n-1$ of the $p = n(n-1)/2$ eigenvalues must be λ_i^2, and the rest $\lambda_i\lambda_j$ for $i < j$. So $\lambda_i^2 = 1$ for all but at most one i. If at least three $\lambda_i = \pm1$ occur, different signs are easily ruled out. The option $n = 4$, $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 \neq \pm1$ is also easy to exclude. Clearly not all $\lambda_i = \pm1$, so all but exactly one are (with the same sign). But then again we must have all $\lambda_i\lambda_j = 1$ for $i \neq j$, which is impossible.

For antisymmetric square the eigenvalues $\lambda_i\lambda_j = 1$ for $i \neq j$ are 1 except one. Let us permute indices j of λ_j so that the exceptional one $-t$ occurs as $\lambda_1\lambda_2$. Then all $\lambda_3\lambda_j$ are equal, so either all λ_j are equal, or equal except one (λ_3). Neither option is possible. □

7.3. D_{2k+1}

Let $n = 2k+1 \geq 5$ be the rank. From Figure 1 it is evident that in order the irrep to lack symmetry, we need to label non-trivially some of the extreme nodes labeled in figure 1 as a_1. They correspond to basic representations called *spinor representations* (see p.351). According to Table 30 page 378, their dimension is 2^{n-1}. We call the nodes a_1 thus below *spinor nodes*.

Now $2^{n-1} \leq (n+1)(n+2)/2$ only if $n = 5$. In this case the only nodes with associated basic representations of dimension at most $(n+1)(n+2)/2 = 21$ are the spinor node, and the other extreme node. So we are left with considering representations dominating

$$\circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!<\begin{matrix}\overset{1}{\circ}\\\circ\end{matrix} \qquad \text{and} \qquad \circ\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!<\begin{matrix}\overset{2}{\circ}\\\circ\end{matrix} \qquad\qquad (16)$$

But by the estimate in lemma 11 their dimensions are at least $3 \cdot 16/2 = 24$. The exact computation using the formula (0.150) gives $N(\phi) = 144$ and 126 resp.

7.4. E_6

In case of H being an E_6 representation, the possible dimensions are $(n+1)(n+2)/2$ for $n \leq 6$. A look at the dimensions of the basic representations of E_6 in Table 30, p.378 shows that only the one marked 27 still fits this bound, and since dimensions do not match, we must try the irrep

$$2 \quad \underset{}{\overset{\circ\!-\!-\!\circ\!-\!-\!\circ\!-\!-\!\circ\!-\!-\!\circ}{}} \qquad . \tag{17}$$

 It (and dominating representations) can be excluded from the estimate in lemma 11.

Originally we were aware, though, just of the strict increase property in the lemma. So we tried to compute the dimension of the irreps of E_6 using the formula (0.153). But this revealed that the preceding calculation of the numbers involved in the formula lacks several explanations and has many errors. So here we provide a correction.

We consider an irrep of E_6 highest weight Λ with a_i in (12) corresponding to nodes of the Dynkin diagram thus:

$$\overset{a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5}{\circ\!-\!-\!\circ\!-\!-\!\circ\!-\!-\!\circ\!-\!-\!\circ} \qquad .$$
$$\underset{a_6}{\circ}$$

First, on p.354, l.-11, λ_5 should be λ_6. On p.355, l.8, one should refer to (0.95) and (0.96) instead of (0.108). On (0.138') of page 355, way may clarify that if we normalize the scalar product so that $||\alpha_i|| = 1$, then $K = 1/12$. Then, on p.378, Table 29, the left node α_6 should be labeled as $\alpha_6 = \lambda_4 + \lambda_5 + \lambda_6 + \lambda$. The positive roots $\Sigma_+(E_6)$ can be then obtained from the list of roots in (0.133) by choosing therein $q > p$ (in the first shape) and the $+$ in either $\pm$ signs (for the second and third shapes).

The formulas on p.358, l.8-10, are almost entirely wrong. The quantities g_k and g on the right are not properly explained, but it is suggestive that the decomposition, similar to (0.145'),

$$g = \sum_{i=1}^{6} g_i \lambda_i + g_0 \lambda$$

for the element g of (14) or (0.140) is meant, where we replaced the g of p.358, l.8-10 by g_0 to avoid confusion. (So g is a vector for us, given in (0.140), and g_0 is a scalar.) We have to assume similarly to (0.146') that

$$\sum_{i=1}^{6} g_i = 0,$$

in order to have formula (0.153) working properly, and we should (with our convention) replace $1/2g$ by $1/2g_0$ therein. The formulas on p.358, l.8-10 should read then

$$l_k = l_6 + \sum_{i=k}^{5} a_i, \qquad\qquad g_k = \frac{7 - 2k}{2} \quad \text{for } k \leq 5;$$

$$l_6 = -\frac{1}{6} \sum_{i=1}^{5} i\, a_i, \qquad\qquad g_6 = -\frac{5}{2};$$

$$l = a_1 + 2a_2 + 3a_3 + 2a_4 + a_5 + 2a_6, \qquad g_0 = 11.$$

To make the presentation more self-contained, we rewrite the dimension formula (0.153) here. With

$$m_k = l_k + g_k, \quad \text{and} \quad m = l + g_0,$$

we have

$$N(\phi) = \frac{m}{g} \prod_{1 \leq p < q \leq 6} \frac{m_p - m_q}{g_p - g_q} \prod_{1 \leq p < q < r \leq 6} \frac{m_p + m_q + m_r + 1/2m}{g_p + g_q + g_r + 1/2g_0}.$$

Regardless of the aforementioned errors, the data for the dimensions of the basic representations of E_6 in Table 30, p.378, are correct. We then also found the exact dimension of the irrep (17) to be 351. (This, and some of the preceding, exact computations of dimension will become helpful below.)

With this the proof of theorem 1 is complete.

8.   Generalizations

After we found our proof of theorem 1, we became aware of the paper by I. Marin [18]. It turns out that our theorem 1 is more or less equivalent to his theorem B and its corollary in §5.1 for groups of type A. His result implies (most of) theorem 1 as follows. In the terminology of our proof, one can conclude from Zariski density (instead of going through Dynkin's list) that $\tilde{H} = SL(p, \mathbb{C})$ when t and q are algebraically independent. (So this is a slightly stronger restriction than ours.) The converse implication is also quite obvious. Marin's result applies also to other generalizations of ρ_n (of his types D and E), and uses an entirely different description of the representation. The proof is quite abstract and consists in looking at the Lie algebra of our H.

In contrast, our proof is more direct and gives some new insight. For example, since the eigenvalue argument can be carried out on $\rho_n(\sigma_1^2)$ instead of $\rho_n(\sigma_1)$, theorem 1 holds by replacing B_n by any subgroup which contains the elements $\beta_{n,k}$ ($k = 2, \ldots, n-1$) in (11), and on which ρ_n is irreducible. Here is a further variation, in which we made also some effort to extract what conditions on t and q are really needed in our arguments.

Theorem 5 Fix an integer $m \neq 0$. Assume q, t with $|t| = |q| = 1$ are chosen so that t, q, tq, tq^2 and tq^n are not roots of unity. Assume the Budney form is definite at q, t, and $G \subset B_n$ is a subgroup as specified below, such that ρ_n is irreducible on G at q, t.

1. If G contains $< \sigma_{2k-1}^{2m} >$ ($k \leq n/2$), and provided tq^3 is not a root of unity when $n = 4$, then $\overline{\rho_n(G)} \simeq U(p)$ (for $p = n(n-1)/2$).

2. If for fixed $a \geq 2$ and l, the group G contains $< \sigma_{ak+l}^{2m} >$ (for all k with $1 \leq ak + l \leq n - 1$), then for n large (in a way dependent on a, but independent on q, t, l or G) we have $\overline{\rho_n(G)} \simeq U(p)$.

Proof. The condition on tq^n is needed to reduce the problem from $U(p)$ to $SU(p)$. The eigenvalue argument remains the same: as long as t is not a root of unity, we can get disposed of symmetry and Kronecker product by looking at $\rho_n(\sigma_1^{2mm'}) \in \hat{H}_0$ for proper m'. The condition on q, tq and tq^2 enters in order to keep lemma 7 working. With these restrictions, the condition on tq^3 is what remains from the second listed assumption in lemma 4, which is needed to adapt the argument for lemma 6. (When $n > 4$, then (5) shows that the eigenvalue 1 occurs too often.) The need to exclude these quantities from being roots of unity (rather than just ± 1) comes again from the possibility that $\hat{H}$ is not connected (i.e. $\hat{H} \neq \hat{H}_0$).

For the second claim a torus (within H) of dimension a positive multiple of the number of braid strands is found from looking at the action of $\rho_n(\sigma_{ak+p}^{2m})$ on subspaces of $v_{ak+p-1,ak+p}$ (where the condition on tq^2 is needed). Such a torus keeps an irrep analysis still manageable. E_6 is relevant only for finitely many n, and D_{2k+1} needs (in order to prevent symmetry) a spinor node marked, with an exponential increase in dimension. Finally for A_n one remarks that any other labeling than the ones we studied would give a dimension of the irrep, which is a polynomial in n of degree > 2. Thus only finitely many n would be relevant.

In the case $a = 2$ of the first claim, we have for a rank-n-group an irrep of dimension $n'(n'-1)/2$ for $n' \leq 2n+1$; in particular the dimension is at most $(2n+1)n$. A similar but slightly more involved discussion in cases, as for the proof of theorem 1, shows that in fact under this weaker condition, still no irreps occur.

To conclude this it is helpful to use the dimension formulas given in the proof of theorem 1 (rather than just the rough estimates). We give just a few details.

For E_6 the only new irrep fitting the dimension bound is
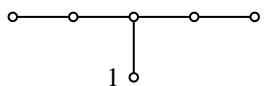
but it is symmetric.

For D_{2k+1}, after applying suitably lemma 11, the only new possibility is the irrep of A_7 obtained from the diagram on the right of (16) by adding two nodes on the left. But from the latter dimension calculated below (16), we conclude that the dimension is > 105.

For A_n, the discussion is slightly lengthier. First, by using the dimension formulas for the basic representations in Table 30, and lemma 11, one sees that one only needs to look at representations where only the leftmost 3 nodes may obtain a non-trivial label. These are discussed case-by-case.

Most of the options were studied already in the proof of theorem 1. We give a little information on the remaining ones, by noticing that for the irreps

$$
\underset{\circ}{\overset{}{}}\overset{1}{\underset{\bullet}{\rule{1cm}{0.4pt}}}\overset{1}{\underset{\bullet}{\rule{1cm}{0.4pt}}}\cdots\underset{\circ}{\rule{0.6cm}{0.4pt}} \qquad \text{and} \qquad \overset{1}{\underset{\circ}{}}\rule{1cm}{0.4pt}\overset{1}{\underset{\circ}{\rule{1cm}{0.4pt}}}\cdots\underset{\circ}{\rule{0.6cm}{0.4pt}}
$$

(which occur in the extended treatment of (15)), the dimensions are $(n+1)\cdot\binom{n+2}{4}$ and $3\binom{n+2}{4}$, resp. (for $n\geq 5$). The irreps

$$
\overset{3}{\underset{\circ}{}}\rule{1cm}{0.4pt}\underset{\circ}{\rule{1cm}{0.4pt}}\cdots\underset{\circ}{\rule{0.6cm}{0.4pt}} = \mathrm{Sym}^3\pi_1 \qquad \text{and} \qquad \underset{\circ}{}\rule{1cm}{0.4pt}\overset{1}{\underset{\circ}{\rule{1cm}{0.4pt}}}\cdots\underset{\circ}{\rule{0.6cm}{0.4pt}} = \textstyle\bigwedge^3\pi_1,
$$

are again (for small n, where the dimension estimate fails) most conveniently ruled out by an eigenvalue argument. $\square$

The following consequence was motivated by a similar result in [18]. Our advantage is that our restrictions of q,t are weaker, and more explicit. (We do not appeal to the result of Crisp-Paris either.)

Corollary 1 Let $n\geq 3$. Assume q,t with $|t|=|q|=1$ are chosen so that t, q, tq, tq^2, tq^3 (latter only for $n=4$), and tq^n are not roots of unity, and the Budney form is definite and ρ_n is irreducible at q,t. Let $m\neq 0$ be any integer. Then we have

$$
\overline{\rho_n(<\sigma_k^{2m}:1\leq k\leq n-1>)}\simeq U(p).
$$

Proof. The argument in the second paragraph of the proof of lemma 7 with $\tau_i=\sigma_i$ explains why the irreducibility of ρ_n implies the one of its restriction to the specified subgroup. $\square$

With such an argument one can treat also the *Hilden subgroup* [10] $H_{2n}\subset B_{2n}$; from the presentation in [25, §5] one exhibits it to contain the elements σ_{2i-1} ($1\leq i\leq n$). For irreducibility one needs a few extra arguments, which we provide.

Proposition 3 Let $n>2$ be even. For q,t as in lemma 2, ρ_n is irreducible on H_n.

Proof. We repeat the proof of lemma 2, until before the conclusion $V=R$. Now, for $q=1$, we have $R=V_1\oplus V_2$, where V_1 is (linearly) generated by $v_{2i-1,2i}$, $i=1,\ldots,n/2$, and V_2 by all the other $v_{i,j}$. It is easily observed that V_k for $k=1,2$ are irreducible over H_n.

If now $\rho_n(t,q)$ are reducible for q converging to 1, then by orthogonal approximation (see [24]) the irrep decomposition of $\rho_n(t,q)$ is of the form $R=V_1(t,q)\oplus V_2(t,q)$, where $V_k(t,q)\to V_k(t,1)=V_k$ for $q\to 1$ in the sense that there are bases that converge vector-wise; in particular $\dim V_k(t,q)=\dim V_k$.

Now again $-tq^2$ is a unique eigenvalue of $\rho_n(t,q)(\sigma_{2i-1})$ with eigenspace spun by $v_{2i-1,2i}$. Since the matrices of $\rho_n(\sigma_{2i-1})$ are conjugate, we see that some $V_k(t,q)$ must contain all $v_{2i-1,2i}$, $i=1,\ldots,n/2$, and so V_1. By convergence we can have only $V_1\subset V_1(t,q)$, and by dimension reasons $V_1=V_1(t,q)$. But it is direct to verify that for $q\neq 1$, V_1 is not an invariant subspace of $\rho_n(t,q)$. $\square$

It should be remarked that *not* necessarily the same q,t as in lemma 2 would do, and that the above indirect argument spoils our control on how q must be to 1, the way we had it in remark 3. Still it seems not worthwhile to enter into technical calculations in order to have this shortcoming removed, and lemma 2 remains at least qualitatively true.

R. Budney has observed irreducibility of $\rho_n\big|_{H_n}$ previously, at least for small n, but it is the lack of written record that motivated us to supply the preceding proposition.

Corollary 2 Let $n \geq 4$ be even. Assume q,t with $|t| = |q| = 1$ are chosen so that t, q, tq, tq^2, tq^3 (if $n = 4$), and tq^n are not roots of unity, t is close to -1, and q is close to 1 depending on t. Then we have $\rho_n(H_n) \simeq U(p)$. $\square$

There seems no principal obstacle to apply our approach to more general Artin groups, if more explicit (matrix) descriptions of the representations are available.

9.   Non-conjugate braids

The final section is devoted to the proof of Theorem 2.

Theorem 2 was obtained first by Shinjo for knots L. However, her method cannot be pleasantly applied to links, and this was our motivation for a different approach in [24]. We extended Shinjo's result, showing theorem 2 when an $n-1$-braid representation of L has a non-scalar Burau matrix. One could hope to further remove braids in the Burau kernel (which exists at least for $n \geq 5$ [2, 17]), replacing ψ_n by the faithful representation ρ_n. This was the origin for our interest in ρ_n in this paper. In contrast, the faithfulness of ρ_n was not essential for theorem 1. (Our approach there was set out to apply also for many values of q,t which are not algebraically independent, and thus for which ρ_n may not be faithful.)

We should now choose some parameters q,t for which ρ_n is faithful. They will have to be close to $(1,-1)$ in the way that will get clear below, but apart from that they should be kept fixed.

Throughout this section, $\beta \in B_{n-1}$ is a fixed non-central braid representation of the link L. It will turn out very helpful to take advantage of our work in [24] and assume, by having dealt with the other cases, that $\psi_{n-1}(\beta)$ is a scalar matrix. We write as

$$C := \{ \alpha\beta\alpha^{-1} : \alpha \in B_{n-1} \}$$

the conjugacy class of β in B_{n-1}. An element in C will typically be written as β'. Such a β' will be regarded also as element of B_n using the inclusion $B_{n-1} \simeq B_{1,n-1} \subset B_n$.

It is known that the center of B_n is generated by the full twist braid $\Delta^2 = (\sigma_1 \dots \sigma_{n-1})^n$.

Lemma 14 Assume for $\gamma \in B_n$, that $\rho_n(\gamma)$ is scalar. Then γ is a power of the full twist braid.

Proof. Scalar matrices are central, and by the faithfulness of ρ_n, so must be then γ. $\square$

A *linear function* f defined on the set $M(p,\mathbb{C})$ of complex $p \times p$ matrices $M = (m_{ij})$ is an expression of the form

$$f(M) = \sum_{i=1}^{p} \sum_{j=1}^{p} a_{ij} m_{ij}, \tag{18}$$

for fixed $a_{ij} \in \mathbb{C}$. We call f a *trace multiple* if $a_{ij} = 0$ and $a_{ii} = a_{jj}$ for $1 \leq i < j \leq p$.

It is well-known that central matrices in $SU(p)$ are scalar and that the trace is a conjugacy invariant. We showed in [24] that, apart from these trivial cases, there are no linear functions of matrices invariant on a conjugacy class.

Proposition 4 ([24]) Assume that $f : M(p,\mathbb{C}) \to \mathbb{C}$ is a linear function, which is not a trace multiple. Let X be a non-central element in $SU(p)$. Then f is not constant on the conjugacy class of X in $SU(p)$ (considered as a subset of $M(p,\mathbb{C})$).

Lemma 15 Assume that $\psi_{n-1}(\beta)$ is a scalar matrix. Then $\mathrm{tr}\,\rho_n(\beta'\sigma_{n-1})$ for $\beta' \in C$ can be expressed as a linear function of $\rho_{n-1}(\beta')$ for $-t,q$ close to 1. Moreover, this linear function is not a trace multiple.

Proof. We assume that $q, -t$ are chosen close to 1, so that ρ_n is unitary.

We note from (1) that $\rho_n\big|_{B_{n-1}}$ preserves the subset $\mathcal{V}_{n-1} = \{ v_{ij} : 1 \leq i < j \leq n-1 \}$. By unitarity, the vectors $v_{i,n}$ for $1 \leq i < n$ can be modified to $\tilde{v}_{i,n}$, so that $\rho_n\big|_{B_{n-1}}$ acts invariantly on the linear span $\mathcal{V}'$ of $\tilde{v}_{i,n}$. We denote by $\tilde{\rho}_n$ the restriction of ρ_n (regarded as a representation of B_{n-1}) to this space $\mathcal{V}'$.

Since we are interested in evaluating the trace, we have the freedom to change basis. In the basis of $\mathcal{V} = \mathcal{V}_{n-1} \cup \mathcal{V}'$ then we have the form (8) (for $k = n - 1$) with $B = 0$.

Next we look at the matrix A in (8). As in lemma 8 and its proof, we noticed that the eigenvalues of $\tilde{\rho}_n(\beta)$ do not depend on t (although $\tilde{\rho}_n(\beta)$ itself would). So let $t = -1$. In this case we use lemma 1 and the standard fact (see e.g. Note 5.7 and above Example 3.2 in [12]) that $\psi_n\big|_{B_{n-1}}$ is the sum of ψ_{n-1} and a trivial representation τ_{n-1}, to conclude that

$$\tilde{\rho}_n = \psi_{n-1} \oplus \tau_{n-1}. \tag{19}$$

By multiplying the matrix of $\tilde{v}_{i,n}$ by a proper unitary matrix independent on t, we can assume that a basis of $\mathcal{V}'$ is chosen w.l.o.g. so that the direct sum in (19) is visible in the block A of (8) for $t = -1$.

Now we assumed that $\psi_{n-1}(\beta)$ is scalar, so all its eigenvalues are the same. But lemma 8 argued that they do not depend on t, so they will be the same also when $t \neq -1$. Since still $\tilde{\rho}_n$ is unitary for the chosen q, t, we see that $\tilde{\rho}_n(\beta)$ is a diagonal matrix independent on t for such q, t, and it is the same matrix $A = \psi_{n-1}(\beta')$ for all $\beta' \in C$.

This means that, in the basis $\mathcal{V}$, the only entries of $\rho_n(\beta')$ that vary with $\beta' \in C$ are those in the block $\rho_{n-1}(\beta')$ in (8). By writing $\rho_n(\sigma_{n-1})$ in the same basis $\mathcal{V}$, we can then express $\mathrm{tr}\,\rho_n(\beta'\sigma_{n-1})$ as a linear combination of entries of $\rho_{n-1}(\beta')$, with coefficients a_{ij} in (18) depending continuously on q, t. (They will involve the entries of $\rho_n(\sigma_{n-1})$ and the scalar in A, which is up to sign a certain power of q.)

To show that this linear combination is not a trace multiple on $\rho_{n-1}(\beta')$ when β' ranges over C, it suffices, by continuity, to look at $q = -t = 1$. Then the action of σ_i is this of permuting the subscripts i and $i+1$ of the (basis) elements $v_{i,j}$ in the formula (1). Clearly σ_{n-1}, exchanging subscripts $n-1$ and n, does not fix (or take to multiples of themselves) all such elements with $1 \le i < j \le n - 1$. □

Theorem 2 follows by combining the previous three statements, theorem 1, and the result in [24].

Proof of theorem 2. Let $\beta \in B_{n-1}$ be a braid representation of L as a non-central braid. Then, by using lemma 14 (and remark 2), we have that $\rho_{n-1}(\beta)$ is not scalar for proper q, t of definite form.

If $\psi_{n-1}(\beta)$ is not scalar, then the claim follows from the work in [24]. So assume that $\psi_{n-1}(\beta)$ is scalar. Then, regarding $B_{n-1} \simeq B_{1,n-1} \subset B_n$, the map $B_{n-1} \to \mathbb{C}$ given by

$$\beta' \mapsto \mathrm{tr}\,\rho_n(\beta'\sigma_{n-1}) \tag{20}$$

is for these q, t linear but not a trace multiple by lemma 15.

By theorem 1, the closure of the ρ_{n-1}-image of the conjugacy class C of β is a $SU(p')$-conjugacy class D with $p' = (n-1)(n-2)/2$. Since (20) is a linear function on C, it can be extended to such a function on D, and in a unique way. This extension is not constant by proposition 4, and D is a connected set. Then this set D cannot contain a dense subset C on which a continuous map (20) takes a finite (or even discrete) value range. □

Thus we prove in fact a bit more; e.g. for proper q, t the set of $|\mathrm{tr}\,\rho_n|$ or $\arg\mathrm{tr}\,\rho_n$ on n-braid representations of L has a closure that contains an interval.

Remark 4 From the perspective of Markov's theorem, it seems more important to construct *irreducible* braids, i.e. such which are *not* conjugate to $\gamma\sigma_{n-1}^{\pm 1}$ for $\gamma \in B_{n-1}$. The examples in the proof of theorem 2 can be easily modified by exchange moves (see [4]) to ones which at least *may* be potentially irreducible. (Lemma 15 needs a slight adaptation.) But this promises no real advance, as long as one cannot *prove* irreducibility. No decent general technique exists to establish this property for non-minimal strand braid representations, except the arguments in [19], which apply in very special cases.

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